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A model of elastoplastic reinforcing media with a tensor damageability parameter☆

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ABSTRACT

A method of constructing models of elastoplastic reinforcing media taking account on the accumulation of defects and plastic incompressibility is proposed. The accumulation of defects is described by a symmetric positive definite second-rank tensor which is included in the number of governing parameters. The model of elastoplastic reinforcing media is formulated in the form of two alternative systems of partial differential equations. One of them is based on the free energy approximation and the other is based on the Gibbs potential approximation. It is shown that these systems satisfy the Hadamard–Legendre condition and inequalities following from the second law of thermodynamics.

Two main directions can be distinguished in fracture mechanics. In one of them, going back in time to Griffith, the processes involved in the origin and development of cracks are investigated.[1] The continuum theory of fracture refers to the other direction in which those states of deformable solids are considered in which the number of different kinds of microdefects is so large that the stress field, the strain field and other physical quantities oscillate rapidly. In this case, the averaged fields of the physical quantities are used and a damage parameter is introduced in order to describe the accumulation of microdefects in a medium. The damageability parameter is included in the number of governing parameters of the model of the behaviour of the medium.

One of the fundamental problems in the continuum theory of fracture involves the establishment of the functional dependence of the damage parameter on the different physical quantities. In order to solve this problem, results obtained when investigating fracture at the microlevel can be invoked: the generation rate and growth of microdefects and the interaction and distribution of microdefects.

In many cases, when the number of micropores is considerably greater than the number of residual microdefects or when microcracks predominate among the microdefects and the distribution of the normals to the surfaces of the microcracks is uniform, a scalar can be introduced as the damage parameter. However, for materials in which the distribution of the normals to the surfaces of the microcracks is non-uniform, the use of a scalar as the damage parameter is insufficient for an adequate description of the accumulation of defects. In the general case, a tensor of a certain rank can be used as the damage parameter. One of the versions of the continuum theory of a medium with cracks[2] is an example of the use of a second rank tensor as the damage parameter. In the last decade there has been increased interest in describing anisotropic fracture processes using models which take account of the accumulation of defects and which use a tensor damage parameter. In part, this interest is associated with the development and investigation of the behaviour of new materials. We mention one of the recent papers[3] concerned with the experimental investigation of the fracture of thin sheets made of aluminium alloys. It is also worth mentioning the review[4] in which several fracture models are considered that use a tensor damage parameter.

1. Basic premises and definitions

Consider an elastoplastic reinforcing media, during the deformation of which the Bauschinger effect is observed, and there is an accumulation of defects. Following the accepted approach in continuum mechanics, we assume that the state of each point mass of a medium is uniquely characterized by some system of parameters of state, and the free energy and the loading function are represented in the form of certain functions of these parameters. Then, from the equations of the first and second laws of thermodynamics and additional assumptions, discussed below, it is possible to derive a system of relations between the state parameters and the generalized thermodynamic forces (or flows) which, after transformation, reduce to a system of partial differential equations.

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It is necessary to impose the following requirements on the latter system:

1°. Any statistically significant experimental data must be satisfactorily approximated by the solutions of this system.
2°. The solutions must satisfy the inequalities which follow from the second law of thermodynamics when account is taken of the irreversibility of the fracture process.
3°. The Cauchy problem with initial data on any uncharacteristic surface in three-dimensional space must be solvable.

Requirement 3° leads to one more condition, the Hadamard–Legendre condition (of strong ellipticity) which, in theories of elasticity and viscoelasticity, is a necessary condition for the Cauchy problem to be solvable in small", that is, for a sufficiently short time interval. It is clear that the same condition will be a necessary condition for the Cauchy problem to be solvable and for constricting local fracture models in which the partial derivatives of the model parameters with respect to the coordinates do not occur in the kinetic equation describing the change in damage. Then, the type of system of differential equations which simulates the motions of such media and the type of system, obtained from it when the kinetic equation is discarded, will be identical. Because of this, the question of the solvability of the Cauchy problem and corresponding mixed problems for the system of equations describing the fracture process reduces to the solvability of these problems in the case of non-linear models which do not take account of fracture. It is necessary to draw attention to the fact that the existing theorems concerning the representation of functions of certain tensor variables do not contain criteria which ensure the Hadamard–Legendre condition is satisfied. It is well known that two sets of variables are used as the governing parameters in mechanics: the stress tensor appears in one (and it is then necessary to proceed from the Gibbs potential) and, in the second, the elastic strain tensor appears (and it is then necessary to proceed from the free energy). We shall subsequently use both sets of governing parameters.

In the case of doubly differentiable convex functions, the Hadamard–Legendre condition follows from the convexity criterion. It is assumed below that the free energy is a function of five arguments, and three of them are tensors (the elastic strain tensor, the residual stress tensor and the damage tensor) and two are scalars (temperature and the reinforcement parameter).

The Hadamard–Legendre condition therefore reduces to the fact that a function which approximates the free energy must be a doubly differentiable convex function with respect to the first argument.

One of the possible methods of approximating the constitutive relations is proposed which enables as to verify the correctness of the above mentioned conditions. Note that this verification leads to non-trivial mathematical calculations. It is worth mentioning that we know of no analysis of the behaviour of a medium of such a kind in the discussion of the majority of models. This, generally speaking, can lead to a state of affairs where certain solutions of a system of equations of a model do not conform with the initial physical prerequisites.

For simplicity, we shall use the Cartesian system of coordinates $\mathbf{x}=(x_1,x_2,x_3)$ and adopt the following notation: ρ is the medium density, T is the temperature, s is the specific entropy, $\mathcal{F}$ is the specific free energy, $\mathbf{v}$ is the velocity vector, $\mathbf{q}$ is the thermal flux vector, $\boldsymbol{\varepsilon}$ is the elastic strain tensor, σ is the stress tensor, $\mathbf{X}$ is the residual stress vector, $\mathbf{e}(\mathbf{v})$ is the strain rate tensor, $\mathbf{e}'$ and $\mathbf{e}''$ are the elastic and plastic strain rate tensors, and $\mathbf{I}$ is the second-rank unit tensor. We shall denote the deviators of the second rank tensors by a superscript D (for example, σ^D is the deviator of the stress tensor), the operation of transposition by the superscript T and the convolution of any two tensors by the symbol $\mathbf{a}\cdot\mathbf{b}$.

In the system of coordinates considered, we shall not distinguish between the matrix of the components of a tensor and the tensor itself.

We will denote the first invariant of the second rank tensor $\mathbf{a}=(a_{ij})$ by $j(\mathbf{a})=a_{11}+a_{22}+a_{33}$, and the scalar product of any second rank tensors $\mathbf{a}$ and $\mathbf{b}$ will be denoted by

$$\mathbf{a}:\mathbf{b}=\mathbf{b}:\mathbf{a}=j\left(\mathbf{a}\cdot\mathbf{b}^T\right)=j\left(\mathbf{a}^T\cdot\mathbf{b}\right)$$

If $\mathbf{c}=(c_{ijkl})$ is a fourth rank tensor and $\mathbf{b}=(b_{ij})$ is a second rank tensor, it is possible to define the second rank tensor $\mathbf{cb}=((cb)_{ij})=(c_{ijkl}b_{kl})$.

Henceforth, the operation of summation is implied over pairs of repeated indices:

1) if the indices indicate components of a tensor, the summation is from 1 to 3;
2) if the index denotes the number of a coefficient and the ordinal number of a fourth rank tensor, the summation is from 1 to 5.
3) if the index denotes the number of a coefficient and the exponent of a second rank tensor, the summation is from 0 to 2.

We shall call fourth rank tensors, the components of which do not change on permutation of the first two (second two) indices and on permutation of the first and second pairs of indices, symmetric tensors, departing from the generally accepted definition. For such tensors, the identity

$$\mathbf{Aa}:\mathbf{b}=\mathbf{a}:\mathbf{Ab}$$

holds, where $\mathbf{a}$ and $\mathbf{b}$ are any symmetric second rank tensors.

We shall call fourth rank tensors $\mathbf{A}$, satisfying the condition

$$\mathbf{Ab}:\mathbf{b}>0 \quad \text{when} \quad \mathbf{b}\neq 0$$

positive definite.

The powers of a second rank tensor $\mathbf{h}$ are defined by the formulae

$$\mathbf{h}^0=\mathbf{I},\quad \mathbf{h}^k=\mathbf{h}\cdot\mathbf{h}^{k-1},\quad k=1,2,\ldots$$

We will next assume that the elastic strains and the convective terms are small. We shall denote differentiation with respect to time by a dot over a vector, tensor or scalar.

We will now consider a deformable solid containing a sufficiently large number of microcracks and take a “physically infinitesimal volume” containing N microcracks. We will denote the jump in the displacement vector at the k-th microcrack by $\Delta_k\mathbf{u}$ and the normal to it by $\mathbf{n}_k$. We introduce the quantities

$$\mathbf{D}_k = \mathbf{n}_k \otimes \Delta_k\mathbf{u} + \Delta_k\mathbf{u} \otimes \mathbf{n}_k$$

The mean value of these quantities is then taken as the damage factor, that is,

$$\mathbf{D} = \frac{1}{N}\sum_{k=1}^{N}\mathbf{D}_k$$

It is obvious that the quantity $\mathbf{D}$ introduced in this way[2] is a second rank tensor.

Remark. The quantity $\mathbf{D}_k$ is independent of the choice of the normal $\mathbf{n}_k$ since

$$\Delta_k\mathbf{u} = \lim_{s\to 0}\left(\mathbf{u}(x + s\mathbf{n}_k) - \mathbf{u}(x - s\mathbf{n}_k)\right)$$

If the opposite vector is taken instead of $\mathbf{n}_k$, the jump $\Delta_k\mathbf{u}$ also changes sign but the quantity $\mathbf{D}_k$ does not change.

The process of the accumulation of a different kind of microdefects (microporesor, microcracks) in a solid, that can be described using the damageability tensor $\mathbf{D}$ which, in the general case is a fourth rank tensor, is commonly referred to as anisotropic fracture in the literature. We shall henceforth confine ourselves to the case when the damage parameter is characterized by a second rank tensor $\mathbf{D} = (D_{ij})$.

2. General propositions of the thermodynamics of irreversible processes

We shall characterize the accumulation of damage by means of a second rank tensor $\mathbf{D}$, which is considered to be symmetric and positive definite. We will assume that $\mathbf{D} = 0(\mathbf{D} > 0)$ corresponds to an undamaged (damaged) state of the medium and the principal values of the tensor $\mathbf{D}$ do not exceed unity. If the largest principal value becomes equal to unity, it becomes impossible to describe the behaviour of the medium within the limits of continuum theory, and it is necessary to introduce additional assumptions such as to assume that macrocracks are formed in the medium, for example, and to specify the law according to which they develop. This stage of the fracture process will not be considered.

In the general case, the accumulation of damage can be characterized not by one but by several parameters of a different physical nature (for example, by parameters which depend on the action of mechanical loads and electromagnetic waves on a body) which leads to the use of tensors of different ranks. This more complex case of fracture is not considered here.

We shall assume that the state of the medium at each point is characterized by the system of parameters ε, T, $\mathbf{X}$, $\varkappa$, $\mathbf{D}$. Here, $\mathbf{X}$ is a second rank tensor and $\varkappa$is a scalar. We shall represent the loading surface in the form of a family of spheres in the six-dimensional space of the stress tensors. The position of the centre of symmetry of a sphere is characterized by the tensor $\mathbf{X}$, and the radius of the sphere is assumed to be a function of the parameter $\varkappa$and, perhaps, the temperature. The parameters $\mathbf{X}$ and $\varkappa$are called the reinforcement parameters. The parameter $\mathbf{D}$ differs from the parameters $\mathbf{X}$ and $\varkappa$in that, for the media considered, the free energy $\mathcal{F}$ decreases during the accumulation of damage as physical considerations show.

We shall then consider, in accordance with the accepted assumption in the theory of elastoplastic media, that the tensor $\mathbf{e}$ can be represented in the form of a sum

$$\mathbf{e} = \mathbf{e}' + \mathbf{e}'' \tag{2.1}$$

We introduce the two inverse second rank tensors

$$\mathbf{h} = \sqrt{(\mathbf{I} - \mathbf{D})}, \quad \mathbf{H} = \mathbf{h}^{-1} \tag{2.2}$$

Taking account of the definition of the tensor $\mathbf{D}$, we conclude from definitions (2.2) that $\mathbf{h}$ and $\mathbf{H}$ are also symmetric, positive definite tensors and $\mathbf{h} = \mathbf{I}$ when $\mathbf{D} = 0$. The principal values of $\mathbf{h}$ are positive and do not exceed unity. Both the tensor $\mathbf{D}$ as well as the tensors $\mathbf{h}$ or $\mathbf{H}$ will be used in describing the fracture process.

The condition that the free energy decreases as damage is accumulated can be written in the form

$$\frac{\partial\mathcal{F}}{\partial\mathbf{D}} : \dot{\mathbf{D}} = \frac{\partial\mathcal{F}}{\partial\mathbf{h}} : \dot{\mathbf{h}} \le 0; \quad \mathcal{F} = \mathcal{F}(\boldsymbol{\varepsilon}, T, \mathbf{X}, \varkappa, \mathbf{D}) \tag{2.3}$$

If $\mathbf{D}$ is a spherical tensor (that is, $\mathbf{D} = d\mathbf{I}$), inequality (2.3) reduces to

$$\frac{\partial\mathcal{F}}{\partial d}\dot{d} \le 0$$

Since it is assumed that d is a non-decreasing function of time, it follows from the last inequality that $\mathcal{F}$ is a decreasing function of the parameter d. A model[5] has been constructed precisely for this case in which the Lamé parameters (or, what is the same thing, the magnitudes of the velocities of the longitudinal or transverse waves) are taken to be decreasing functions of the parameter d.

The relation

$$\rho T\dot{s} = -T\nabla \cdot \left(\frac{\mathbf{q}}{T}\right) + Q$$

where Q is the entropy production rate (uncompensated heat), follows from the equation of the second law of thermodynamics

$$Q = -\frac{\nabla T\cdot\mathbf{q}}{T} + \boldsymbol{\sigma}:\mathbf{e}'' - \left(\frac{\partial\mathcal{F}}{\partial\boldsymbol{\varepsilon}} - \boldsymbol{\sigma}\right):\mathbf{e}' - \rho\left(\frac{\partial\mathcal{F}}{\partial T} - s\right)\dot{T} - \rho\frac{\partial\mathcal{F}}{\partial\mathbf{X}}:\dot{\mathbf{X}} - \rho\frac{\partial\mathcal{F}}{\partial\varkappa}\dot{\varkappa} - \rho\frac{\partial\mathcal{F}}{\partial\mathbf{D}}:\dot{\mathbf{D}}$$

and the outies–Kirahhoff–Neumann energy equation, that which follows from the first law of thermodynamics. The equality $\mathbf{e}' = d\varepsilon/dt$, which holds in the case of small deformations, has been used in deriving this relation.

According to the second law of thermodynamics, the quantity Q in the last relations is equal to zero in the case of reversible processes and non-negative for any irreversible processes. It is assumed in the theory of elastoplastic media that the quantity Q is independent of the tensor $\mathbf{e}'$ and the rate of change of temperature T.

In this case, the equation of the second law of thermodynamics reduces to the form

$$\rho T\dot{s} = -\nabla\cdot\mathbf{q} + \boldsymbol{\sigma}:\mathbf{e}'' - \rho\frac{\partial\mathcal{F}}{\partial\mathbf{X}}:\dot{\mathbf{X}} - \rho\frac{\partial\mathcal{F}}{\partial\varkappa}\dot{\varkappa} - \rho\frac{\partial\mathcal{F}}{\partial\mathbf{D}}:\dot{\mathbf{D}} \tag{2.4}$$

and the tensor $\boldsymbol{\sigma}$ and the specific entropy are defined using the formulae

$$\boldsymbol{\sigma} = \rho\frac{\partial\mathcal{F}}{\partial\boldsymbol{\varepsilon}},\quad s = -\frac{\partial\mathcal{F}}{\partial T} \tag{2.5}$$

In order to obtain the constitution relations for the other system of governing parameters $\boldsymbol{\sigma}$, T, $\mathbf{X}$, $\varkappa$, $\mathbf{D}$, we introduce the Gibbs potential using the formula

$$\rho\Psi = \rho\mathcal{F} - \boldsymbol{\sigma}:\boldsymbol{\varepsilon}$$

In this equality $\varepsilon = \varepsilon(\sigma, T, \mathbf{X}, \varkappa, \mathbf{D})$ is the solution of the system

$$\boldsymbol{\sigma} = \rho\frac{\partial\mathcal{F}(\boldsymbol{\varepsilon}, T, \mathbf{X}, \varkappa, \mathbf{D})}{\partial\boldsymbol{\varepsilon}}$$

for specified σ ; T, $\mathbf{X}$, $\varkappa$, $\mathbf{D}$.

The function $\Psi = \Psi(\sigma, T, \mathbf{X}, \varkappa, \mathbf{D})$ is identical, apart from the sign, with the Legendre transform for the free energy. The relations

$$\boldsymbol{\varepsilon} = -\rho\frac{\partial\Psi}{\partial\boldsymbol{\sigma}},\quad s = -\frac{\partial\Psi}{\partial T},\quad \frac{\partial\Psi}{\partial\mathbf{h}} = \frac{\partial\mathcal{F}}{\partial\mathbf{h}},\quad \frac{\partial\Psi}{\partial\mathbf{D}} = \frac{\partial\mathcal{F}}{\partial\mathbf{D}} \tag{2.6}$$

therefore hold. Taking account of the last two equalities of (2.6), the dissipation condition in the case of fracture (2.3) can be written as

$$\frac{\partial\Psi}{\partial\mathbf{D}}:\dot{\mathbf{D}} = \frac{\partial\Psi}{\partial\mathbf{h}}:\dot{\mathbf{h}} \le 0 \tag{2.7}$$

Relations (2.5) (or (2.6)) enable us to determine the stress tensor $\boldsymbol{\sigma}$ (or the elastic strain tensor $\boldsymbol{\varepsilon}$) and the entropy s if the explicit representation of the free energy as a function of the state parameters ε, T, $\mathbf{X}$, $\varkappa$, $\mathbf{D}$ (or the representation of the Gibbs potential as a function of the state parameters σ, T, $\mathbf{X}$, $\varkappa$, $\mathbf{D}$) is known. The quantities $\dot{\mathbf{D}}$, $\dot{\mathbf{X}}$, $\varkappa$, which have still by no means been determined, occur in Eq. (2.4). It is therefore necessary to find representations of the free energy and the quantities $\dot{\mathbf{D}}$, $\dot{\mathbf{X}}$ such that Eq. (2.4) and the inequality $Q \ge 0$ are satisfied.

This problem is not a mathematical problem in the strict sense since the enumerated conditions are insufficient to find $\dot{\mathcal{F}}$, $\dot{\mathbf{D}}$, $\dot{\mathbf{X}}$, $\dot{\varkappa}$ as a function of the parameters ε, T, $\mathbf{X}$, $\varkappa$, $\mathbf{D}$. On the other hand, these functions must be such that it is possible to approximate the experimental data with some accuracy. It therefore only makes sense to speak of the approximation of the constitutive relations with a specified accuracy. We next consider the problem of designing a continuum model of the fracture of a material with a damage tensor from precisely this point of view.

3. Approximation of the free energy (the Gibbs potential)

We will now consider the motion of a medium in the case of small temperature deviations $\theta = T - T_0$ from a certain given value. In order to establish in what form the free energy and the Gibbs potential must be represented in this case, it is necessary to take account of two experimental facts. As a rule, for structural steels, the entropy production rate depends only slightly on the rate of change in the parameters $\varkappa$and $\mathbf{X}$ and, in the elastic region, the relation between the stress and elastic strain tensors is independent of $\varkappa$and $\mathbf{X}$. It is possible to take account of these facts assuming the free energy to be a doubly differentiable function and the following relations

$$\frac{\partial^2\mathcal{F}}{\partial T\partial\mathbf{X}} = 0,\quad \frac{\partial^2\mathcal{F}}{\partial\boldsymbol{\varepsilon}\partial\mathbf{X}} = 0,\quad \frac{\partial^2\mathcal{F}}{\partial T\partial\varkappa} = 0,\quad \frac{\partial^2\mathcal{F}}{\partial\boldsymbol{\varepsilon}\partial\varkappa} = 0$$

held. In this case, the free energy is represented in the form of a sum of two functions

$$\mathcal{F}(\boldsymbol{\varepsilon}, T, \mathbf{X}, \varkappa, \mathbf{h}) = \mathcal{F}_0(\mathbf{X}, \varkappa, \mathbf{h}) + \mathcal{F}_1(\boldsymbol{\varepsilon}, T, \mathbf{h})$$

For simplicity, we will next consider the case when the function $\mathcal{F}_0(\mathbf{X}, \mathbf{h})$ is independent of $\varkappa$, since taking account of this dependence does not lead to any fundamental difficulties. In the case of small increments in $\boldsymbol{\varepsilon}$ and θ, the free energy can be approximated by a second degree Taylor polynomial. The approximate equality

$$\mathcal{F}(\boldsymbol{\varepsilon},T,\mathbf{X},\mathbf{h}) \cong \mathcal{F}_0(\mathbf{X},\mathbf{h}) + s_0\theta + \frac{1}{\rho}\boldsymbol{\sigma}_0 : \boldsymbol{\varepsilon} + \frac{1}{2}c_\varepsilon\theta^2 + \frac{1}{\rho}\boldsymbol{\beta} : \boldsymbol{\varepsilon}\theta + \frac{1}{2\rho}\mathbf{c}\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} \tag{3.1}$$

then holds. Here,

$$s_0 = s_0(\mathbf{h}) = \frac{\partial\mathcal{F}_1}{\partial T}(\boldsymbol{\Sigma}),\ \boldsymbol{\sigma}_0 = \boldsymbol{\sigma}_0(\mathbf{h}) = \rho\frac{\partial\mathcal{F}_1}{\partial\boldsymbol{\varepsilon}}(\boldsymbol{\Sigma}),\ c_\varepsilon = c_\varepsilon(\mathbf{h}) = \frac{\partial^2\mathcal{F}_1}{\partial T^2}(\boldsymbol{\Sigma})$$
$$\boldsymbol{\beta} = \boldsymbol{\beta}(\mathbf{h}) = \rho\frac{\partial^2\mathcal{F}_1}{\partial\boldsymbol{\varepsilon}\partial T}(\boldsymbol{\Sigma}),\quad \mathbf{c} = \mathbf{c}(\mathbf{h}) = \rho\frac{\partial^2\mathcal{F}_1}{\partial\boldsymbol{\varepsilon}^2}(\boldsymbol{\Sigma}),\quad \boldsymbol{\Sigma} = (0,T_0,\mathbf{h}) \tag{3.2}$$

where $\mathbf{c}$ is the elastic moduli tensor. Note that the coefficient $c_\varepsilon = c_\varepsilon(\mathbf{h})$ is connected with the specific heat capacity at constant deformations $\tilde{c}_\varepsilon$ by the relation

$$\tilde{c}_\varepsilon = -Tc_\varepsilon$$

We shall assume that $\sigma_0(\mathbf{h}) = 0$ when $\mathbf{h} = \mathbf{I}$. Taking account of relations (3.1) and (3.2), the dissipation condition (2.3) can be written in the form

$$\rho\overset{\triangledown}{\mathcal{F}}_0 + \rho\dot{s}_0\theta + \dot{\boldsymbol{\sigma}}_0 : \boldsymbol{\varepsilon} + \frac{1}{2}\dot{c}_\varepsilon\theta^2 + \dot{\boldsymbol{\beta}} : \boldsymbol{\varepsilon}\theta + \frac{1}{2}\dot{\mathbf{c}}\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} \le 0;\quad \overset{\triangledown}{\mathcal{F}}_0 = \frac{\partial\mathcal{F}_0}{\partial\mathbf{h}} : \dot{\mathbf{h}} \tag{3.3}$$

Since $\boldsymbol{\varepsilon}$ and θ are independent variables, the two inequalities

$$\rho\overset{\triangledown}{\mathcal{F}}_0 + \rho\dot{s}_0\theta + \frac{1}{2}\dot{c}_\varepsilon\theta^2 \le 0;\quad \rho\overset{\triangledown}{\mathcal{F}}_0 + \dot{\boldsymbol{\sigma}}_0 : \varepsilon + \frac{1}{2}\dot{\mathbf{c}}\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} \le 0 \tag{3.4}$$

follow from inequality (3.3).

Taking account of the fact that $\mathcal{F}_0(\mathbf{X}, \mathbf{h}) = \mathcal{F}(0, T_0, \mathbf{X}, \mathbf{h})$, we obtain from condition (2.3) that the inequality

$$\overset{\triangledown}{\mathcal{F}}_0 \le 0 \tag{3.5}$$

must be satisfied.

Since the physical nature of the tensor $\mathbf{h}$ has not been taken into account, it is impossible to make any judgment regarding the signs of the remaining terms in inequality (3.2). However, when account is taken of inequality (3.5), it is possible to show that $\delta_T(\mathbf{h}) > 0$ and $\delta_\varepsilon(\mathbf{h}) > 0$ may be found such that inequalities (3.4) will be satisfied if $|\theta| \le \delta_T$, $\varepsilon : \varepsilon \le \delta_\varepsilon^2$.

We next find the representation for the Gibbs potential in the case when the free energy has the form (3.1). Taking account of the definition of the Gibbs potential, after some reduction we reduce the first relation of (2.5) and formula (3.1) to the relation

$$\Psi(\boldsymbol{\sigma},T,\mathbf{X},\varkappa,\mathbf{h}) = \mathcal{F}_0(\mathbf{X},\mathbf{h}) + s_0\theta + \frac{1}{\rho}\boldsymbol{\varepsilon}_0 : \boldsymbol{\sigma} - \frac{1}{2\rho}\mathbf{a}\boldsymbol{\sigma} : \boldsymbol{\sigma} + \frac{1}{\rho}\boldsymbol{\alpha} : (\boldsymbol{\sigma} - \boldsymbol{\sigma}_0)\theta + \frac{c_\sigma}{2}\theta^2 \tag{3.6}$$

Here $\mathbf{a} = \mathbf{a}(\mathbf{h}) = (a_{ijkm}(\mathbf{h}))$ is a tensor which is the inverse of the tensor $\mathbf{c} = \mathbf{c}(\mathbf{h})$, that is, such that the equality $\mathbf{a}:\mathbf{c} = \mathbf{I}\cdot\mathbf{I}$ is satisfied (the operation$\circ$ is defined in the second paragraph after formula (3.7)), the tensor $\mathbf{a}$ is symmetric and positive definite: $\mathbf{a}\sigma : \sigma > 0$, $\alpha = \mathbf{a}\beta$ is a symmetric second rank tensor and $\varepsilon_0 = \varepsilon_0(\mathbf{h}) = \mathbf{a}\sigma_0(\mathbf{h})$, $c_\sigma = c_\varepsilon - \rho^{-1}\ \alpha:\beta$. Note that the coefficient $c_\sigma = c_\sigma(\mathbf{h})$ is connected with the heat capacity at constant stresses $\tilde{c}_\sigma$ by the relation

$$\tilde{c}_\sigma = -Tc_\sigma$$

We shall consider media which are isotropic in the undamaged state and for which the free energy in this state has the form

$$\mathcal{F}_* = \mathcal{F}_*(\boldsymbol{\varepsilon},T,\mathbf{X},\varkappa) = F_0\mathbf{X} : \mathbf{X} + s_0\theta + \frac{1}{2}c_\varepsilon\theta^2 + \frac{3}{\rho}\alpha_0 Kj(\boldsymbol{\varepsilon})\theta + \frac{\lambda}{2\rho}j^2(\boldsymbol{\varepsilon}) + \frac{\mu}{\rho}\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon};\ K = \lambda + \frac{2}{3}\mu\ \ (3.7) \tag{3.7}$$

Here $F_o > 0$, $s_0 < 0$, $c_\varepsilon < 0$ are constants, α_0 is the volume expansion coefficient, $\lambda > 0$ and $\mu > 0$ are the Lamé parameters and K is the bulk modulus.

We will now consider the question of in which form the function $\mathcal{F}_0$ and the tensors $\boldsymbol{\beta}$ and $\mathbf{c}$ can be given in order that the free energy, defined by relation (3.1), is a convex function in the variable $\boldsymbol{\varepsilon}$, and the dissipation of mechanical energy during inelastic deformation and fracture is positive. In order to answer this question, one of the methods of approximating the function $\mathcal{F}_0$ and the tensors $\boldsymbol{\beta}$ and $\mathbf{c}$ will be proposed below. A slight mathematical digression is required for this.

We will introduce a definition. Consider the linear space of second rank tensors. We will define a binary operation which matches any two tensors $\mathbf{a}$ and $\mathbf{b}$ with a fourth rank tensor $\mathbf{a} \circ \mathbf{b}$ according to the formula

$$(\mathbf{a} \circ \mathbf{b})\boldsymbol{\tau} = \frac{1}{4}\left(\mathbf{a}\cdot\boldsymbol{\tau}\cdot\mathbf{b} + \mathbf{b}\cdot\boldsymbol{\tau}\cdot\mathbf{a} + \mathbf{a}\cdot\boldsymbol{\tau}^T\cdot\mathbf{b} + \mathbf{b}\cdot\boldsymbol{\tau}^T\cdot\mathbf{a}\right)$$

Here $\boldsymbol{\tau}$ is any second rank tensor. It can be verified that a tensor $\mathbf{a} \circ \mathbf{b}$ defined in this way is symmetric:

$$(\mathbf{a} \circ \mathbf{b})_{ijkm} = (\mathbf{a} \circ \mathbf{b})_{jikm} = (\mathbf{a} \circ \mathbf{b})_{kmij}$$

The components of a tensor $\mathbf{a}\circ\mathbf{b}$ are represented in the form

$$(\mathbf{a}\circ\mathbf{b})_{ijkm}=\frac{1}{4}(a_{ik}b_{mj}+a_{im}b_{kj}+b_{ik}a_{mj}+b_{im}a_{kj})$$

It is obvious that $\mathbf{a}\circ\mathbf{b}=\mathbf{b}\circ\mathbf{a}$. If $\tau=\tau^T$, then

$$(\mathbf{a}\circ\mathbf{b})\boldsymbol{\tau}=\frac{1}{2}(\mathbf{a}\cdot\boldsymbol{\tau}\cdot\mathbf{b}+\mathbf{b}\cdot\boldsymbol{\tau}\cdot\mathbf{a})$$

When $\mathbf{a}=\mathbf{b}$, the tensor $\mathbf{a}\circ\mathbf{b}$ acts according to the rule $(\mathbf{a}\circ\mathbf{a})\tau=\mathbf{a}\cdot\tau\cdot\mathbf{a}$. For example, $(\mathbf{I}\circ\mathbf{I})\tau=\tau$.

We now consider the system of fourth rank tensors

$$\mathbf{C}^0=\mathbf{I}\circ\mathbf{I},\ \ \mathbf{C}^1=\mathbf{h}\circ\mathbf{h},\ \ \mathbf{C}^2=\mathbf{I}\circ\mathbf{h},\ \ \mathbf{C}^3=\mathbf{I}\circ\mathbf{h}^2,\ \ \mathbf{C}^4=\mathbf{h}\circ\mathbf{h}^2,\ \ \mathbf{C}^5=\mathbf{h}^2\circ\mathbf{h}^2,\ \ \mathbf{C}^6=\mathbf{h}\otimes\mathbf{h}$$

The tensors $\mathbf{C}^k=\mathbf{C}^k(\mathbf{h})(k=0,1,\ldots,5)$ generate the set of all tensors of the form $\mathbf{h}^p\circ\mathbf{h}^q(p,q=1,2\ldots)$. This follows from the Hamilton–Cayley identity.

Using this system, we introduce the quadratic forms

$$\varphi_0(\boldsymbol{\varepsilon},\mathbf{h})=f(\mathbf{h})j^2(\boldsymbol{\varepsilon}),\quad f(\mathbf{h})=f(j_1(\mathbf{h}),\ j_2(\mathbf{h}),\ j_3(\mathbf{h}))\geq 0$$

$$f(\mathbf{I})=1,\quad \frac{\partial f}{\partial j_k}\geq 0,\quad j_k(\mathbf{h})=j(\mathbf{h}^k),\quad k=1,2,3 \tag{3.8}$$

$$\varphi_k(\boldsymbol{\varepsilon},\mathbf{h})=\frac{1}{2}\mathbf{C}^k(\mathbf{h})\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon},\quad k=1,2,\ldots,6 \tag{3.9}$$

which depend on the tensor $\mathbf{h}$. It follows from the definition of the forms $\varphi_k=\varphi_k(\varepsilon,\mathbf{h})$ that, when $\mathbf{h}=\mathbf{I}$, the equalities

$$\varphi_0(\boldsymbol{\varepsilon},\mathbf{I})=j^2(\boldsymbol{\varepsilon}),\quad \varphi_k(\boldsymbol{\varepsilon},\mathbf{I})=\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon},\quad k=1,2,\ldots,6 \tag{3.10}$$

will hold.

The forms $\varphi_k(k=0,\ldots,5)$ are positive definite as function of the first argument. In the case of the form φ_0, this is an obvious consequence of definition (3.8). The following fact is required in order to show that this assertion holds when $k=1,\ldots,5$.

Lemma 1. For any symmetric second rank tensor $\mathbf{z}$, the quadratic form

$$\Phi(\mathbf{z},\mathbf{a},\mathbf{b})=j(\mathbf{a}\cdot\mathbf{z}\cdot\mathbf{b}\cdot\mathbf{z}) \tag{3.11}$$

is positive definite with respect to $\mathbf{z}$:

$$\Phi(\mathbf{z},\mathbf{a},\mathbf{b})>0,\quad \mathbf{z}\neq 0$$

where $\mathbf{a}$ and $\mathbf{b}$ are symmetric positive definite matrices.

In fact, unique symmetric positive definite matrices $\mathbf{a}'$ and $\mathbf{b}'$ exist such that the equalities

$$\mathbf{a}'\cdot\mathbf{a}'=\mathbf{a},\quad \mathbf{b}'\cdot\mathbf{b}'=\mathbf{b}$$

are satisfied.

The right-hand side of expression (3.11) can therefore be transformed as follows:

$$j(\mathbf{a}\cdot\mathbf{z}\cdot\mathbf{b}\cdot\mathbf{z})=j(\mathbf{a}'\cdot\mathbf{a}'\cdot\mathbf{z}\cdot\mathbf{b}'\cdot\mathbf{b}'\cdot\mathbf{z})=j(\mathbf{a}'\cdot\mathbf{z}\cdot\mathbf{b}'\cdot\mathbf{b}'\cdot\mathbf{z}\cdot\mathbf{a}')=j\left(\mathbf{a}'\cdot\mathbf{z}\cdot\mathbf{b}'\cdot(\mathbf{a}'\cdot\mathbf{z}\cdot\mathbf{b}')^T\right)>0$$

We will now return to the proof that the forms φ_k are positive definite. Any of them, when account is taken of relation (3.9), can be written in the form

$$\varphi_k(\boldsymbol{\varepsilon},\mathbf{h})=(\mathbf{h}^m\circ\mathbf{h}^n)\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon}=j(\mathbf{h}^m\cdot\boldsymbol{\varepsilon}\cdot\mathbf{h}^n\cdot\boldsymbol{\varepsilon}),\quad m,n=0,1,2$$

Since $\mathbf{h}^k>0(k=0,1,2)$, it follows from Lemma 1 that the right-hand side of the last relation is positive when $\varepsilon\neq 0$. This also means that the forms φ_k are positive definite and, consequently, convex in the variable $\boldsymbol{\varepsilon}$.

We will now show that the quadratic forms φ_k satisfy the condition

$$\frac{\partial\varphi_k}{\partial\mathbf{D}}\leq 0,\quad k=0,1,\ldots,5 \tag{3.12}$$

With this aim, we now prove the following assertion.

Lemma 2. Suppose **a** and **b** are symmetric, positive definite matrices and $\mathbf{a}>0$. If the inequality $\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}>0$ holds then $\mathbf{b}>0$ also.

Proof. Suppose $\mathbf{c}=\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}$. Without loss of generality, it can be assumed that **b** is a diagonal matrix $\mathbf{b}=\mathrm{diag}(b_1,b_2,\ldots b_n)$. The elements of the matrix **c** then have the form

$$c_{kj}=\left(b_k+b_j\right)a_{kj}$$

It follows from the conditions $\mathbf{c}>0$ that $c_{kk}>0(k=1,2,\ldots,n)$. Then,

$$b_k=\frac{c_{kk}}{2a_{kk}}>0, k=1,2,\ldots,n$$

Taking account of the criterion for the symmetric matrices to be positive definite, we conclude that $\mathbf{b}>0$.

We will now verify inequality (3.12) for the forms φ_0 and φ_4. The verification for the remaining forms is carried out in a similar manner. In fact, taking account of the relation for the variation $\delta\mathbf{D}$, which follows from definition (2.2),

$$\delta\mathbf{D}=-(\mathbf{h}\cdot\delta\mathbf{h}+\delta\mathbf{h}\cdot\mathbf{h})=-(\mathbf{I}\circ\mathbf{h})\cdot\delta\mathbf{h}=-\delta\left(\mathbf{h}^2\right)$$

the variation of the form φ_4 can be written in the form

$$\delta\varphi_4(\boldsymbol{\varepsilon},\mathbf{h})=\mathbf{C}^4(\mathbf{h}+\delta\mathbf{h})\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon}-\mathbf{C}^4(\mathbf{h})\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon}=(\mathbf{Y}_1+\mathbf{Y}_2):\delta\mathbf{D}$$

Here $\mathbf{Y}_2=-\varepsilon\cdot\mathbf{h}\cdot\varepsilon$, and the tensor $\mathbf{Y_1}$ is connoted with the tensor $\varepsilon\cdot\mathbf{h}^2\cdot\varepsilon$ by the relation

$$\left(\boldsymbol{\varepsilon}\cdot\mathbf{h}^2\cdot\boldsymbol{\varepsilon}\right):\delta\mathbf{h}=\mathbf{Y}_1:\delta\mathbf{D}$$

We derive the equality

$$-\boldsymbol{\varepsilon}\cdot\mathbf{h}^2\cdot\boldsymbol{\varepsilon}=\mathbf{h}\cdot\mathbf{Y}_1+\mathbf{Y}_1\cdot\mathbf{h}$$

from this last relation, taking account of the definition of the tensor $\mathbf{I}\circ\mathbf{h}$. Since $\mathbf{h}\geq 0$ and the left-hand side of the last equality is a negative definite tensor, it follows from Lemma 2 that $\mathbf{Y}_1\leq 0$. Taking account of the fact that $\mathbf{Y}_2\leq 0$, we therefore obtain

$$\frac{\partial\varphi_4}{\partial\mathbf{D}}=\mathbf{Y}_1+\mathbf{Y}_2\leq 0$$

In order to prove inequality (3.12) when $k=0$, it is sufficient to prove that the relation

$$\frac{\partial f}{\partial\mathbf{D}}\leq 0,\ \text{ if }\ \frac{\partial f}{\partial j_k}\geq 0,\ k=1,2,3$$

holds.

In fact, an increment in the function f can be written in the form

$$\delta f=f(\mathbf{h}+\delta\mathbf{h})-f(\mathbf{h})=f(\mathbf{D}+\delta\mathbf{D})-f(\mathbf{D})=\left(\frac{\partial f}{\partial j_1}\mathbf{Y}_1-\frac{\partial f}{\partial j_2}\mathbf{I}+\frac{\partial f}{\partial j_3}(\mathbf{Y}_2-\mathbf{h})\right):\delta\mathbf{D}$$

where $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are determined from the equalities

$$-\mathbf{I}=\mathbf{Y}_1\cdot\mathbf{h}+\mathbf{h}\cdot\mathbf{Y}_1,\quad -\mathbf{h}^2=\mathbf{Y}_2\cdot\mathbf{h}+\mathbf{h}\cdot\mathbf{Y}_2$$

Since $\mathbf{h}\geq 0$ and the left-hand sides of the last two equalities are negative definite tensors, it follows from Lemma 2 that $\mathbf{Y}_1\leq 0$, $\mathbf{Y}_2\leq 0$. The inequality

$$\frac{\partial f}{\partial\mathbf{D}}=\frac{\partial f}{\partial j_1}\mathbf{Y}_1-\frac{\partial f}{\partial j_2}\mathbf{I}+\frac{\partial f}{\partial j_3}(\mathbf{Y}_2-\mathbf{h})\leq 0$$

therefore holds and inequality (3.12) follows from this when $k=0$.

We will now show that the quantity $\partial\varphi_6/\partial\mathbf{D}$ is not a second rank tensor of fixed sign. We prove this assertion by contradiction and assume, for example, that $\partial\varphi_6/\partial\mathbf{D}\leq 0$. Then, from Lemma 1, putting $\mathbf{z}=\mathbf{I}$, we obtain the inequality

$$\frac{\partial\varphi_6}{\partial\mathbf{D}}:\dot{\mathbf{D}}\leq 0$$

On the other hand, from the definition of the form φ_6 we have

$$\frac{\partial\varphi_6}{\partial\mathbf{D}}:\dot{\mathbf{D}}=\frac{\partial\varphi_6}{\partial\mathbf{h}}:\dot{\mathbf{h}}=\frac{1}{2}\left(\dot{\mathbf{h}}\otimes\mathbf{h}+\mathbf{h}\otimes\dot{\mathbf{h}}\right)\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon}=\frac{1}{2}(\mathbf{h}:\boldsymbol{\varepsilon})\left(\dot{\mathbf{h}}:\boldsymbol{\varepsilon}\right)$$

The right-hand side of the last relation can take both positive and negative values.

Actually, we take the tensor $\boldsymbol{\varepsilon}$ in the form $\varepsilon = \mathrm{diag}(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ and choose $\varepsilon_1,\varepsilon_2,\varepsilon_3$ such that the inequalities

$$\varepsilon_k h_{kk} \geq 0, \quad \varepsilon_k \dot{h}_{kk} \leq 0$$

are satisfied.

The inequality

$$(\mathbf{h} : \boldsymbol{\varepsilon})(\dot{\mathbf{h}} : \boldsymbol{\varepsilon}) = (\varepsilon_m h_{mm})(\varepsilon_n \dot{h}_{nn}) \leq 0$$

then holds. On the other hand, when the tensor $\boldsymbol{\varepsilon}$ is positive definite and account is taken of the inequalities $\mathbf{h} > 0$, $\dot{\mathbf{h}} < 0$, according to Lemma 1 and putting $\mathbf{z} = \mathbf{I}$, we have

$$(\mathbf{h} : \boldsymbol{\varepsilon})(\dot{\mathbf{h}} : \boldsymbol{\varepsilon}) \geq 0$$

We have thereby proved that the inequality $\partial\varphi_6/\partial\mathbf{D} \leq 0$ does not hold. It can similarly be proved that the inequality $\partial\varphi_6/\partial\mathbf{D} \geq 0$ also does not hold.

Hence, the quantity $\partial\varphi_6/\partial\mathbf{D}$ is not of fixed sign which does not permit using the form φ_6 in the approach proposed here to the approximation of the constitutive relations.

Properties (3.10) and (3.12) of the quadratic forms (3.8) and (3.9) as well as the representation of the free energy, corresponding to the undamaged state of the medium, enable us to assume that the function $\mathcal{F}$ can be approximated in the form

$$\mathcal{F}(\boldsymbol{\varepsilon},T,\mathbf{X},\mathbf{h}) = F_0 \mathbf{dX} : \mathbf{X} + \frac{1}{\rho}\sigma_0 : \boldsymbol{\varepsilon} + s_0\theta + \frac{1}{2}c_\varepsilon\theta^2 + \frac{1}{\rho}\beta : \boldsymbol{\varepsilon}\theta + \frac{\lambda}{2\rho}\mathbf{g}\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} + \frac{\mu}{\rho}j^2(\boldsymbol{\varepsilon})f \tag{3.13}$$

Here,

$$\mathbf{d} = \mathbf{d}(\mathbf{h}) = r_k\mathbf{C}^k, \quad \mathbf{g} = \mathbf{g}(\mathbf{h}) = s_k\mathbf{C}^k, \quad \boldsymbol{\beta} = \boldsymbol{\beta}(\mathbf{h}) = 3\alpha_0 K\beta_k\mathbf{h}^k$$

$$r_k = r_k(\mathbf{h}) \geq 0, \quad \frac{\partial r_k}{\partial\mathbf{D}} \leq 0, \quad s_k = s_k(\mathbf{h}) \geq 0, \quad \frac{\partial s_k}{\partial\mathbf{D}} \leq 0 \quad \sum r_k(\mathbf{I}) = \sum s_k(\mathbf{I}) = \sum\beta_k(\mathbf{I}) = 1$$

and the function f is defined by relations (3.8).

Note that the right-hand side of relation (3.13) can be obtained from relation (3.7) if we make the change of variables

$$\mathbf{X} : \mathbf{X} \to \mathbf{dX} : \mathbf{X}, \quad \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} \to \mathbf{g}\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}, \quad j^2(\boldsymbol{\varepsilon}) \to j^2(\boldsymbol{\varepsilon})f \tag{3.14}$$

on the right-hand side of the latter equation.

Comparing relations (3.1) and (3.13), we obtain an explicit expression for the tensor $\mathbf{c}$

$$\mathbf{c} = \lambda\mathbf{g} + 2\mu f\,\mathbf{I} \circ \mathbf{I}$$

We will now prove some properties of the function $\mathcal{F}$:

1) convexity with respect to the variable $\boldsymbol{\varepsilon}$;
2) $\mathcal{F}(\varepsilon, T, \mathbf{X}, \kappa, \mathbf{I}) = \mathcal{F}_*(\varepsilon, T, \mathbf{X}, \kappa)$;
3) the inequality

$$\mathbf{Y} \triangleq \frac{\partial\mathcal{F}}{\partial\mathbf{D}} \leq 0 \tag{3.15}$$

holds.

Property 1 follows form the definition of the tensor $\mathbf{g}$ and the convexity of the forms $\varphi_k (k = 0, 1, \ldots, 5)$. Property 2 follows from the definition of the tensors $\mathbf{d}$ and $\mathbf{g}$ and the function $\mathcal{F}$.

In order to simplify the calculations, we will prove inequality (3.15) for isothermal processes. Note that the quadratic forms $\mathbf{dX} : \mathbf{X}$, $\mathbf{g}\varepsilon : \varepsilon$ are linear combinations of the quadratic forms (3.9) with positive coefficients, for which the inequality (3.12) holds. Taking account of relation (3.8) and the constraints on the coefficients r_k and s_k, the inequalities

$$\frac{\partial(\mathbf{dX} : \mathbf{X})}{\partial\mathbf{D}} \leq 0, \quad \frac{\partial(\mathbf{g}\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon})}{\partial\mathbf{D}} \leq 0, \quad \frac{\partial f}{\partial\mathbf{D}} \leq 0$$

can be obtained, from which inequality (3.15) follows. Inequality (3.15) for non-isothermal processes in the case of small increments in θ can be proved in a similar way. To do this, it is necessary to take account of the quantity

$$\frac{\partial s_0}{\partial\mathbf{D}}\theta, \quad \frac{\partial^2 c_\varepsilon}{\partial\mathbf{D}^2}\theta^2, \quad \frac{\partial\boldsymbol{\beta}}{\partial\mathbf{D}} : \boldsymbol{\varepsilon}\theta$$

in the expression for the derivative $\partial\mathcal{F}/\partial\mathbf{D}$, the contribution of which can be made as small as desired by an appropriate choice of the increment θ. Inequality (3.15) will play a key role in the approximation of the law for the change in the damageability tensor.

The quantities $\sigma_0, s_0, c_\varepsilon, r_k, s_k (k = 0, \ldots, 5)$, $\beta_m (m = 0,1,2)$ occur in relation (3.13). Here, they are assumed to be known. The problem of determining of these quantities leads to the solution of the corresponding inverse problems. (As a rule, inverse problems are found to be ill-posed and special methods have been developed for solving them.) However, inverse problems can only be formulated when the full system of differential equations, which the governing parameters must satisfy, is derived.

If an approximation of the constitutive relations is required not only in the neighbourhood $T=T_0$, $\varepsilon=0$ but, also, in a certain domain $\Omega\subset\mathbb{R}^6$ in the space of the stresses, it is possible to proceed in the following manner. We divide the domain Ω into a finite number of subdomains Ω_k, $\Omega=\cup\Omega_k (k=1, 2, \ldots, N)$ and assume that Ω_1 is the neighbourhood of an unloaded state, where the approximation of the function $\mathcal{T}$ is given by formula (3.13). In each of the subdomains Ω_k, we choose a certain value $\sigma_k\in\Omega_k$ of the stress tensor, denote the value of the elastic strain tensor corresponding to it by $\boldsymbol{\varepsilon}_k$ and assume that the increment $\Delta_k\varepsilon=\varepsilon-\varepsilon_k$ is small within each subdomain. The values of the variables $\mathbf{X}$, $\boldsymbol{\kappa}$, $\mathbf{D}$ are fixed, and we then approximate the free energy in each of the subdomains $\Omega_2,\ldots,\Omega_N$ by a second-degree Taylor polynomial in the increments θ and $\Delta_k\varepsilon$ in a similar manner to (3.1). After this, we repeat all the arguments, starting from those concerning relation (3.7). We then arrive at the approximation of the free energy in the domain Ω.

An approximation of the Gibbs potential corresponding to the unloaded state of the medium can be constructed if we express the quantities $\mathbf{a}$, $\boldsymbol{\alpha}$ and c_σ in terms of $\mathbf{c}$, $\boldsymbol{\beta}$ and c_ε which are already known. We will now consider the question of the form in which the tensor $\mathbf{a}$ should be given if nothing is known about the tensor $\mathbf{c}$.

In order to answer this question, it is necessary to specify the representation of the Gibbs potential corresponding to the unloaded state of the medium. It is possible to proceed from the expression

$$\rho\tilde{\Psi}=\rho F_0\mathbf{X}:\mathbf{X}+\rho s_0\theta+\frac{\rho}{2}c_\sigma\theta^2+\alpha_0 j(\boldsymbol{\sigma})\theta-\frac{1+\nu}{2E}\left(\boldsymbol{\sigma}:\boldsymbol{\sigma}-\frac{\nu}{1+\nu}j^2(\boldsymbol{\sigma})\right) \tag{3.16}$$

corresponding to the representation of the free energy in the form (3.7). In relation (3.16), ν is Poisson's ratio and E is Young's modulus. The expression in parentheses is a difference between positive quantities, and it is impossible in this case to establish whether this difference is positive. It is then impossible to conclude whether the Gibbs potential is a concave function with in the parameter $\boldsymbol{\sigma}$.

In order to overcome this difficulty, we use the identity

$$\boldsymbol{\sigma}:\boldsymbol{\sigma}\equiv\boldsymbol{\sigma}^D:\boldsymbol{\sigma}^D+\frac{1}{3}j^2(\boldsymbol{\sigma})$$

Expression (3.16) is then transformed into another expression which is more convenient for analysis

$$\rho\tilde{\Psi}=\rho F_0\mathbf{X}:\mathbf{X}+\rho s_0\theta+\frac{\rho}{2}c_{\boldsymbol{\sigma}}\theta^2+\alpha_0 j(\boldsymbol{\sigma})\theta-\frac{1+\nu}{2E}\left(\boldsymbol{\sigma}^D:\boldsymbol{\sigma}^D+\frac{1-2\nu}{3(1+\nu)}j^2(\boldsymbol{\sigma})\right) \tag{3.17}$$

Since $0\le\nu<1/2$, both terms in parentheses in relation (3.17) are positive. It is then possible to repeat the arguments which led to the approximation (3.13) for the free energy and, by analogy with algorithm (3.14), replace the variables on the right-hand side of (3.17)

$$\boldsymbol{\sigma}^D:\boldsymbol{\sigma}^D\to\mathbf{G}\boldsymbol{\sigma}^D:\boldsymbol{\sigma}^D,\quad j^2(\boldsymbol{\sigma})\to j^2(\boldsymbol{\sigma})F(\mathbf{h})$$

where the function F obeys the same relations as the function f, which occurs in relation (3.8) but with the single exception that $\partial F/\partial j_k\le 0$. We then obtain a representation of the Gibbs potential in the form

$$\begin{aligned}\rho\Psi&=\rho F_0\mathbf{dX}:\mathbf{X}+\rho s_0\theta+\frac{\rho}{2}c_{\boldsymbol{\sigma}}\theta^2+\\&+\boldsymbol{\sigma}:\boldsymbol{\varepsilon}_0+\boldsymbol{\alpha}:(\boldsymbol{\sigma}-\boldsymbol{\sigma}_0)\theta-\frac{1+\nu}{2E}\left(\mathbf{G}\boldsymbol{\sigma}^D:\boldsymbol{\sigma}^D+\frac{1-2\nu}{3(1+\nu)}j^2(\boldsymbol{\sigma})F\right)\end{aligned} \tag{3.18}$$

Here,

$$\mathbf{G}=\mathbf{G}(\mathbf{H})=\tilde{s}_k(\mathbf{H})\mathbf{C}^k(\mathbf{H}),\quad \tilde{s}_k\ge 0,\quad \frac{\partial\tilde{s}_k}{\partial\mathbf{D}}\ge 0,\quad \sum\tilde{s}_k(\mathbf{I})=1,$$
$$k=0,\ldots,5;\quad \boldsymbol{\alpha}=\boldsymbol{\alpha}(\mathbf{h})=\alpha_0\tilde{\beta}_k\mathbf{h}^k,\quad \sum\tilde{\beta}_k(\mathbf{I})=1$$

where $\tilde{\beta}_k=\tilde{\beta}_k(\mathbf{h})$ are given functions.

Note that the tensor function $\mathbf{G}$ depends explicitly on the tensor $\mathbf{H}$ but not on the tensor $\mathbf{h}$. In this connection if the tensor $\mathbf{h}$ were to be its argument, this would lead to the inequality $\partial\Psi/\partial\mathbf{D}>0$, whereas it follows from relations (2.6) and (3.17) that the inverse inequality holds.

Comparing relations (3.6) and (3.18), we obtain an explicit expression for the tensor $\mathbf{a}$

$$\mathbf{a}=\frac{1+\nu}{E}\left(\mathbf{G}-\frac{2}{3}\mathbf{GI}\otimes\mathbf{I}+\left(G_0+\frac{1-2\nu}{3(1+\nu)}F\right)\mathbf{I}\otimes\mathbf{I}\right),\ G_0=\frac{1}{9}\sum_{k=1}^{3}G_{kkkk}$$

Note the following properties of the function Ψ

1) concavity with respect to the variable $\boldsymbol{\sigma}$;
2) $\Psi(\sigma, T, \mathbf{X}, \kappa, \mathbf{I})=\tilde{\Psi}(\sigma, T, \mathbf{X}, \kappa)$;
3) $\partial\Psi/\partial\mathbf{D}\le 0$.

Property 1 follows from the convexity of the quadratic forms $\varphi_k(\sigma^D, \mathbf{H}) = \mathbf{C}^k(\mathbf{H})\sigma^D : \sigma^D$. Property 2 follows from the equalities

$$\varphi_k\left(\boldsymbol{\sigma}^D, \mathbf{I}\right) = \boldsymbol{\sigma}^D : \boldsymbol{\sigma}^D, \quad k = 1, \ldots, 5, \quad F(\mathbf{I}) = 1, \quad \boldsymbol{\alpha}(\mathbf{I}) = \alpha_0 \mathbf{I}$$

In order to show that Property 3 holds, we will consider isothermal processes for simplicity. It is then sufficient to verify that inequalities (3.12) hold. In fact, the equalities

$$\frac{\partial \mathbf{H}}{\partial \mathbf{h}} = -\mathbf{H} \circ \mathbf{H}, \quad \frac{\partial \mathbf{D}}{\partial \mathbf{h}} = -\mathbf{I} \circ \mathbf{h} \tag{3.19}$$

follow from the definitions of the tensors $\mathbf{h}$ and $\mathbf{H}$. Hence, on replacing the derivatives $\partial \mathbf{H}/\partial \mathbf{h}$ and $\partial \mathbf{D}/\partial \mathbf{h}$ in the equality

$$\frac{\partial \mathbf{H}}{\partial \mathbf{h}}\left(\frac{\partial \varphi_k}{\partial \mathbf{H}}\right) = \frac{\partial \mathbf{D}}{\partial \mathbf{h}}\left(\frac{\partial \varphi_k}{\partial \mathbf{D}}\right)$$

with the right-hand sides of equalities (3.19), we obtain

$$\mathbf{H} \cdot \frac{\partial \varphi_k}{\partial \mathbf{H}} \cdot \mathbf{H} = \mathbf{h} \cdot \frac{\partial \varphi_k}{\partial \mathbf{D}} + \frac{\partial \varphi_k}{\partial \mathbf{D}} \cdot \mathbf{h}$$

The inequalities $\partial \varphi_k / \partial \mathbf{H} \geq 0 (k = 1, \ldots, 5)$ follow from definition (3.9) if the tensor $\mathbf{h}$ in it is replaced by the tensor $\mathbf{H}$. The correctness of Property 3 then follows from Lemma 2.

We now consider the special case of approximation (3.18) when

$$\mathbf{G} = \mathbf{H} \circ \mathbf{H}, \quad F(j_1, j_2, j_3) = \frac{1}{1 + \eta(j_2 - 3)}, \quad 0 < \eta < 1, \quad \frac{\partial F}{\partial j_1} = \frac{\partial F}{\partial j_3} = 0, \quad \frac{\partial F}{\partial j_2} < 0$$

In this case, the Gibbs potential can be represented in the form

$$\rho \Psi = \rho F_0 \, \mathbf{dX} : \mathbf{X} + \rho s_0 \theta + \frac{\rho}{2} c_\sigma \theta^2 + \boldsymbol{\sigma} : \boldsymbol{\varepsilon}_0 + \boldsymbol{\alpha} : (\boldsymbol{\sigma} - \boldsymbol{\sigma}_0)\theta -$$
$$-\frac{1+\nu}{2E}\left(\left(\mathbf{H} \cdot \boldsymbol{\sigma}^D\right) : \left(\mathbf{H} \cdot \boldsymbol{\sigma}^D\right) + \frac{1 - 2\nu}{3(1+\nu)(1 - \eta j(\mathbf{D}))} j^2(\boldsymbol{\sigma})\right)$$

If we discard the first five terms in the last expression, we arrive at the representation of the Gibbs potential, apart from sign and notation, which was postulated earlier[3,7] for the isothermal case.

4. The law of variation of the tensor D

The a priori determination of the law of variation of the tensor $\mathbf{D}$ is impossible. However, it is of interest to analyse the possible representations of the rate of change of the tensor $\mathbf{D}$ as a function of the governing parameters.

In the case of elastoplastic media, the set of admissible values of the stress tensor is the family of closed convex sets $\mathbb{K}_{x æ h}$, which depends on $\mathbf{X}$,æ,$\mathbf{h}$. As a rule, the sets $\mathbb{K}_{x æ h}$ are specified by the inequalities

$$f(\boldsymbol{\sigma}, \mathbf{X}, æ, \mathbf{h}) \leq 0 \tag{4.1}$$

To simplify the notation, the dependence of the loading function f on the temperature in the inequality is not shown. At internal points of the set $\mathbb{K}_{\mathbf{x}æ\mathbf{h}}$, where the strict inequality $f < 0$ holds, the medium behaves elastically, that is, the equalities

$$\dot{\mathbf{X}} = \dot{\mathbf{h}} = \mathbf{e}'' = 0, \quad \dot{æ} = 0 \tag{4.2}$$

are satisfied. On the boundary of the set $\mathbb{K}_{\mathbf{x}æ\mathbf{h}}$, where the equality $f = 0$ holds, plastic flow can occur, that is, the inequalities

$$\dot{\mathbf{X}} \neq 0, \quad \dot{\mathbf{h}} \neq 0, \quad \dot{æ} \neq 0, \quad \mathbf{e}'' \neq 0$$

are satisfied.

In order to design a model of a reinforcing elastoplastic medium, it is necessary to express the quantities $\dot{\mathbf{X}}$, $\dot{\mathbf{h}}$, $\dot{æ}$, $\mathbf{e}''$ in terms of σ, $\mathbf{X}$, $\mathbf{h}$, æ, $(\sigma \in \mathbb{K}_{\mathbf{X}æ\mathbf{h}})$ so that inequality (4.1) and the conditions enumerated above hold. The normality principle formulated earlier[8,9] is used to design models of reinforcing elastoplastic media and includes Drucker's postulate[6] as a special case.

In order to formulate the normality principle, we introduce the variables

$$x_1 = \boldsymbol{\sigma}, \quad x_2 = \rho \frac{\partial \mathcal{F}}{\partial \mathbf{X}}, \quad x_3 = \rho \frac{\partial \mathcal{F}}{\partial \mathbf{D}}, \quad x_4 = \rho \frac{\partial \mathcal{F}}{\partial \boldsymbol{\kappa}} \tag{4.3}$$

and consider the set of quantities (x_1, x_2, x_3, x_4) as vectors in the linear space $\mathbb{R}^{19}$. We rewrite the inequality (4.1) in the variables $x = (x_1, x_2, x_3, x_4)$

$$f(\boldsymbol{\sigma}, \mathbf{X}, æ, \mathbf{h}) = \varphi(x_1, x_2, x_3, x_4) \leq 0 \tag{4.4}$$

We shall assume that $\varphi(x)$ is a convex function such that $\varphi(0) \leq 0$. Inequality (4.4) then specifies a closed convex set $\mathbb{K}_B$ in the space $\mathbb{R}^{19}$ which contains the origin of coordinates. We will call the boundary of the set $\mathbb{K}$ the loading surface and denote it by $\partial \mathbb{K}$.

The normality principle holds under these assumptions:

1) if the state of the medium is such that the vector $x \in \mathbb{K} \backslash \partial\mathbb{K}$, then the medium behaves elastically, that is, conditions (4.2) are satisfied;
2) if the state of the medium is such that the vector $x \in \partial\mathbb{K}$, then the vector $p = (p_1, p_2, p_3, p_4)$ with the components

$$p_1 = \mathbf{e}'', \quad p_2 = -\dot{\mathbf{X}}, \quad p_3 = -\dot{\mathbf{D}}, \quad p_4 = -\dot{æ}$$

must belong to the normal cone $N_K(x)$, that is, the variational inequality

$$0 \geq \sum_{j=1}^{4} p_j (y_j - x_j), \quad \forall \boldsymbol{y} = (y_1, y_2, y_3, y_4) \in \mathbb{K} \tag{4.5}$$

must be satisfied. The multiplication operation in the first three terms of inequality (4.5) is a convolution of second rank tensors.

According to the assumption $y = 0 \in \mathbb{K}$, it therefore follows from the definition of the vector $\mathbf{p}$ and relations (4.3) and (4.5) that the dissipation of mechanical energy is positive. Note that the variables **X**, æ, **D** are equally justified in the case considered and plastic flow and fracture are the manifestations of a single deformation process. Consequently, condition (2.3) is superfluous in this case.

If the loading surface is smooth, the following relation holds

$$N_K(\mathbf{x}) = \{\mathbf{p} \mid \mathbf{p} = \lambda \nabla\varphi, \lambda > 0\}; \quad \nabla\varphi = \left(\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, \frac{\partial\varphi}{\partial x_3}, \frac{\partial\varphi}{\partial x_4}\right)$$

In this case, the constitutive relations

$$\mathbf{e}'' = \lambda\frac{\partial\varphi}{\partial x_1}, \quad \dot{\mathbf{X}} = -\lambda\frac{\partial\varphi}{\partial x_2}, \quad \dot{\mathbf{D}} = -\lambda\frac{\partial\varphi}{\partial x_3}, \quad \dot{æ} = -\lambda\frac{\partial\varphi}{\partial x_4}, \quad \lambda \geq 0, \quad \varphi \leq 0, \quad \lambda\varphi = 0 \tag{4.6}$$

follow from the normality principle.

If, as a result of the change of variables $(\sigma, \mathbf{X}, æ, \mathbf{h}) \to (x_1, x_2, x_3, x_4)$, the function f turns out to be non-convex, it is impossible to use the normality principle in the form formulated above. In this case, the quantities $\mathbf{e}''$, $\dot{\mathbf{X}}$, $\dot{\mathbf{D}}$ and $\dot{æ}$ have to be determined from additional considerations, which will be presented for the quantities $\mathbf{e}''$, $\dot{\mathbf{X}}$ and $\dot{æ}$ in the following section.

We will now consider the question as to how to approximate the law of variation of the tensor **D** if the loading function φ is not convex. For simplicity, we will confine ourselves to the case of isothermal processes. Taking account of inequality (3.15), in order to design a consistent model of a continuous medium it is sufficient to specify the law of variation of the quantity **D** in the form

$$\dot{\mathbf{D}} = \mathbf{f}_D(\boldsymbol{\sigma}, \mathbf{D}, \mathbf{e}'') \tag{4.7}$$

In order for relation (2.3) to be satisfied, $\mathbf{f}_D$ must be a symmetric, positive definite vector function.

The other important property of the function $\mathbf{f}_D$ is its homogeneity. It is assumed in the theory of plasticity that the dissipation function must be a homogeneous function of the first order in to the argument $\mathbf{e}''$. It has already been mentioned that the dissipation resulting from fracture is represented in the form of two cofactors, one of which depends and the other does not depend on $\mathbf{e}''$. It then follows from relation (4.7) that the tensor function $\mathbf{f}_D$ is positively homogeneous in the argument $\mathbf{e}''$, that is, the relation

$$\mathbf{f}_D(\boldsymbol{\sigma}, \mathbf{D}, \lambda\mathbf{e}'') = \lambda\mathbf{f}_D(\boldsymbol{\sigma}, \mathbf{D}, \mathbf{e}''), \quad \forall\lambda > 0 \tag{4.8}$$

must be satisfied. In this case, the tensor **D** can be considered as a further reinforcement parameter.

In order to determine the tensor function $\mathbf{f}_D$ in general form, which satisfies the above mentioned conditions, we introduce the tensor function

$$|\mathbf{a}| = \sqrt{(\mathbf{a} \cdot \mathbf{a})} = \mathbf{P}^T \operatorname{diag}(|a_1|, |a_2|, |a_3|)\mathbf{P}$$

Here, **a** is any symmetric tensor, a_1, a_2, a_3 are its principal values and **P** is a matrix, the columns of which are formed by the coordinates of the eigenvectors of the matrix **a**. The tensor |**a**| is positive definite and its principal values $|a|_k$ satisfy the equalities

$$|a|_k = |a_k|$$

The tensor function $\mathbf{f}_D$ can then be represented in the form

$$\mathbf{f}_D(\boldsymbol{\sigma}, \mathbf{D}, \mathbf{e}'') = p_k\left(|\mathbf{e}''|^{1/2} \cdot (-\mathbf{Y})^k \cdot |\mathbf{e}''|^{1/2}\right) + q_k\boldsymbol{\sigma}^k \cdot |\mathbf{e}''| \cdot \boldsymbol{\sigma}^k, \quad k = 0,1,2 \tag{4.9}$$

Here, $p_k = p_k(\boldsymbol{\sigma}, \mathbf{D}) \leq 0$, $q_k = q_k(\boldsymbol{\sigma}, \mathbf{D}) \leq 0$ are known functions.

It is obvious that formula (4.9) defines a symmetric, positive definite vector function which satisfies condition (4.8).

It is possible to reduce the number of arbitrary functions used in defining the tensor **D** to three if it is assumed that an additional condition holds: the tensor $\dot{\mathbf{D}}$ is co-axial with the tensor $\mathbf{e}''$ and its principal values $R_k (k = 1, 2, 3)$ are specified using three functions f_k

$$f_k(\boldsymbol{\sigma}, \mathbf{D}, \lambda\mathbf{e}'') = \lambda f_k(\boldsymbol{\sigma}, \mathbf{D}, \mathbf{e}''), \quad \lambda \geq 0, \quad R_k = f_k(\boldsymbol{\sigma}, \mathbf{D}, \mathbf{e}'')$$

We will consider two special cases.

Case 1. Suppose all the functions f_k are identical: $f_k = f(\sigma, \mathbf{D}, \mathbf{e}'') > 0$); The tensor $\dot{\mathbf{D}}$ is then a spherical tensor:

$$\dot{\mathbf{D}} = f\mathbf{I}$$

Case 2. If

$$f_k = q_0|\mathbf{e}''|_k, \quad k = 1,2,3, \quad q_0 = q_0(\boldsymbol{\sigma}, \mathbf{D}) \geq 0,$$

then

$$\dot{\mathbf{D}} = q_0|\mathbf{e}''| \tag{4.10}$$

if we put

$$p_0 = p_1 = p_2 = q_1 = q_2 = 0$$

in relation (4.9).

Note that the representation of the tensor $\dot{\mathbf{D}}$ in the form (4.10) can also be obtained from relation (4.9). Relation (4.10) is identical apart from the notation, to the law for the variation in the tensor $\mathbf{D}$ postulated earlier[7].

We will now consider a version when the right-hand side of relation (4.7) is independent of $\mathbf{e}''$. We will confine ourselves to an analysis of isothermal processes. We recall that the tensor $\mathbf{Y}$ (it is defined in relation (3.15)) is negative definite, and it is therefore possible to use the tensors $\mathbf{Y}$ and $\mathbf{h}$ to approximate the function $\mathbf{f}_D$ in the form

$$\mathbf{f}_D(\boldsymbol{\sigma}, \mathbf{D}) = \mathbf{q}\mathbf{Y} + \mathbf{p}\mathbf{h}; \quad \mathbf{q} = \mathbf{q}(\mathbf{h}) = q_k \mathbf{h}^k \circ \mathbf{h}^k, \quad \mathbf{p} = \mathbf{p}(\mathbf{Y}) = p_k(\mathbf{Y})^k \circ (\mathbf{Y})^k, \quad k = 0,1,2 \tag{4.11}$$

Here, $\mathbf{p}$ and $\mathbf{q}$ are fourth rank tensors. It follows from expression (4.11) that the function $\mathbf{f}_D$ is represented in the form of a linear combination of six tensors. Generally speaking, it is possible to select other systems of linearly independent, positive definite tensors, which differ from the system used in relation (4.11) to approximate the function $\mathbf{f}_D$.

Note that, when $\mathbf{p}(\mathbf{Y}) = 0$, it follows from relation (4.11) that the function for the dissipation resulting from fracture is a quadratic form in the variable $\mathbf{Y}$, which corresponds to the traditional representation of this function in the thermodynamics of non-equilibrium processes

As previously, the number of functions used to define the tensor $\dot{\mathbf{D}}$ can be reduced from six to three. In order to do this, it is necessary to assume in addition that the tensors $\dot{\mathbf{D}}$ and $\mathbf{Y}$ are coaxial and the principal values of the tensor $\dot{\mathbf{D}}$ are specified using the three functions $f_k(\sigma, \mathbf{D}) \geq 0$ ($k = 1,2,3$). We now consider two special cases:

1) if all the function f_k are equal, then $\dot{\mathbf{D}}$ is a spherical tensor;
2) if

$$f_k(\boldsymbol{\sigma}, \mathbf{D}) = f y_k, \quad y_k \geq 0, \quad f \geq 0$$

and y_k ($k = 1,2,3$) are the principal values of the tensor $\mathbf{Y}$, then

$$\dot{\mathbf{D}} = f\mathbf{Y}$$

This last relation also follows from the equality (4.11) if we put $\mathbf{p}(\mathbf{Y}) = 0$, $q_0 = f$, $q_1 = q_2 = 0$.

The first $I_D = j(\mathbf{D})$, second $II_D = j(\mathbf{D}\cdot\mathbf{D})$ and third $III_D = \det(\mathbf{D})$ invariants of the damageability tensor do not decrease during the deformation process. Actually, taking account of the inequality $\mathbf{f}_D \geq 0$, we have

$$\dot{I}_D = j(\dot{\mathbf{D}}) = j(\mathbf{f}_D) \geq 0, \quad \dot{II}_D = j(\dot{\mathbf{D}}\cdot\mathbf{D} + \mathbf{D}\cdot\dot{\mathbf{D}}) = 2\mathbf{f}_D : \mathbf{D} \geq 0, \quad \dot{III}_D = (\det(\mathbf{D}))^{\bullet} = \det(\mathbf{D})\mathbf{f}_D : \mathbf{D}^{-1} \geq 0$$

5. Approximation of $\mathbf{e}''$, $\dot{\mathbf{X}}$, $\dot{æ}$

We shall consider media for which the loading function f_*, corresponding to the unloaded state, is represented in the form

$$f_*(\boldsymbol{\sigma}, T, \mathbf{X}, æ) = (\boldsymbol{\sigma} - \mathbf{X})^D : (\boldsymbol{\sigma} - \mathbf{X})^D - \mathfrak{K}_*(æ, T) \tag{5.1}$$

and the plastic deformations satisfy the incompressibility condition

$$j(\mathbf{e}'') = 0$$

In relation (5.1), $\Re_*$ is an arbitrary decreasing continuous function. Then, taking the assumptions made into account and following the approach proposed for approximating the free energy, we arrive at an approximation of the loading function for describing the damaged state of the medium in the form

$$f(\boldsymbol{\sigma}, T, \mathbf{X}, æ, \mathbf{h}) = \frac{1}{2}\mathbf{b}(\mathbf{h})(\boldsymbol{\sigma} - \mathbf{X})^D : (\boldsymbol{\sigma} - \mathbf{X})^D - \mathfrak{K}(æ, T, \mathbf{h}) \tag{5.2}$$

Here,

$$\mathbf{b}(\mathbf{h}) = q_k \mathbf{C}^k(\mathbf{h}), \quad q_k = q_k(\boldsymbol{\sigma}, \mathbf{h}) \geq 0, \quad \sum_k q_k(\boldsymbol{\sigma}, \mathbf{I}) = 1$$

and $\Re$ is an arbitrary continuous non-decreasing function with respect to the first argument. It is obvious that a loading function defined in this way is convex with respect to the set of variables $\boldsymbol{\sigma}$ and $\mathbf{X}$.

In order to record the constitutive relations for the tensors $\mathbf{e}''$ and $\dot{\mathbf{X}}$, we introduce the variable

$$\boldsymbol{\eta} = \rho\frac{\partial\mathcal{F}}{\partial\mathbf{X}} = \rho F_0\mathbf{d}\mathbf{X}$$

and rewrite relation (5.2) in the variables σ, T, η, $\boldsymbol{\kappa}$, $\mathbf{h}$

$$\varphi(\boldsymbol{\sigma},T,\boldsymbol{\eta},æ,\mathbf{h}) = \frac{1}{2}\mathbf{b}(\mathbf{h})\left(\boldsymbol{\sigma}-\frac{1}{\rho F_0}\mathbf{d}^{-1}\boldsymbol{\eta}\right)^D:\left(\boldsymbol{\sigma}-\frac{1}{\rho F_0}\mathbf{d}^{-1}\boldsymbol{\eta}\right)^D - \mathfrak{K}(æ,T,\mathbf{h}) \tag{5.3}$$

Since $\mathbf{d}$ is a positive definite fourth rank tensor, the inverse tensor $\mathbf{d}^{-1}$ exists. For simplicity, we shall henceforth the function $\Re$ which is independent of the tensor $\mathbf{h}$ since taking account of this dependence does not lead to fundamental difficulties. Therefore, $\Re_* = \Re$.

We shall consider the set of all pairs $(\boldsymbol{\sigma}, \boldsymbol{\eta})$ as the linear space $\mathbb{R}^{12}$. In the space $\mathbb{R}^{12}$, we separate out the set $\mathbb{K}_{æ\mathbf{h}T}$ depending on æ, $\mathbf{h}$, T of those pairs of $(\boldsymbol{\sigma}, \boldsymbol{\eta})$ which satisfy the inequality

$$\varphi(\boldsymbol{\sigma},T,\boldsymbol{\eta},æ,\mathbf{h}) \le 0$$

Since the function φ is convex in to the variables $(\boldsymbol{\sigma}, \boldsymbol{\eta})$, the set $\mathbb{K}_{æ\mathbf{h}T}$ is also convex in $\mathbb{R}^{12}$. By analogy with the normality principle, we therefore assume that the equalities

$$\mathbf{e}'' = \dot{\mathbf{X}} = 0$$

hold for the internal points $(\sigma, \eta) \in \mathbb{K}_{æ\mathbf{h}T}$.

If a pair $(\boldsymbol{\sigma}, \boldsymbol{\eta})$ belongs to the boundary of the set $\mathbb{K}_{æ\mathbf{h}T}$, we assume that the value of $(\mathbf{e}'', -\dot{\mathbf{X}})$ belongs to the normal cone $N_K(\sigma, \eta)$ to the set $\mathbb{K}_{æ\mathbf{h}T}$.

If the set $\mathbb{K}_{æ\mathbf{h}T}$ has a smooth boundary, the conditions formulated above are equivalent to the following:

if $\varphi < 0$, then $\mathbf{e}'' = \dot{\mathbf{X}} = 0$),
if $\varphi = 0$, then

$$\mathbf{e}'' = \lambda\frac{\partial\varphi}{\partial\boldsymbol{\sigma}}, \quad -\dot{\mathbf{X}} = \lambda\frac{\partial\varphi}{\partial\eta}, \quad \lambda \ge 0, \text{ when } \overset{\triangledown}{f} > 0 \tag{5.4}$$

or

$$\mathbf{e}'' = \dot{\mathbf{X}} = 0, \text{ when } \overset{\triangledown}{f} \le 0 \tag{5.5}$$

Here,

$$\overset{\triangledown}{f} = f_\sigma : \dot{\boldsymbol{\sigma}} + f_T\dot{T} + \varsigma f_h : \dot{\mathbf{h}}; \quad f_\sigma = \frac{\partial f}{\partial\boldsymbol{\sigma}}, \quad f_T = \frac{\partial f}{\partial T}, \quad f_h = \frac{\partial f}{\partial\mathbf{h}} \tag{5.6}$$

and $\varsigma = 1$ for media for which the tensor $\dot{\mathbf{D}}$ is independent of $\mathbf{e}''$, and $\varsigma = 0$ for media for which the tensor $\dot{\mathbf{D}}$ depends on $\mathbf{e}''$.

We now calculate $\partial\varphi/\partial\sigma$ and $\partial\varphi/\partial\eta$, taking account of relation (5.3) and obtain

$$\mathbf{e}'' = \lambda\left(\mathbf{b}(\boldsymbol{\sigma}-\mathbf{X})^D\right)^D, \quad \dot{\mathbf{X}} = \frac{\lambda}{\rho F_0}\mathbf{d}^{-1}\left(\mathbf{b}(\boldsymbol{\sigma}-\mathbf{X})^D\right)^D \tag{5.7}$$

According to the first of these equalities, the condition of plastic incompressibility is satisfied. Moreover, the relation

$$\dot{\mathbf{X}} = \frac{\mathbf{d}^{-1}\mathbf{e}''}{\rho F_0} \tag{5.8}$$

follows from equalities (5.7) which, when $\mathbf{h} = \mathbf{I}$, takes the form

$$\dot{\mathbf{X}} = \frac{\mathbf{e}''}{\rho F_0}$$

and, apart from the notation, is identical to the law for the change in the tensor $\mathbf{X}$ proposed by Prager.[10]

The inequality

$$\boldsymbol{\sigma} : \mathbf{e}'' - \rho\frac{\partial\mathcal{F}}{\partial\mathbf{X}} : \dot{\mathbf{X}} \ge 0 \tag{5.9}$$

follows from relations (5.4) and (5.5).

In fact, taking account of relations (5.4) and (5.8), the left-hand side of inequality (5.9) can be transformed to the form

$$\boldsymbol{\sigma} : \mathbf{e}'' - \rho\frac{\partial \mathcal{F}}{\partial \mathbf{X}} : \dot{\mathbf{X}} = \boldsymbol{\sigma} : \mathbf{e}'' - \mathbf{dX} : \mathbf{d}^{-1}\mathbf{e}'' = \lambda(\boldsymbol{\sigma} - \mathbf{X}) : \left(\mathbf{b}(\boldsymbol{\sigma} - \mathbf{X})^{D}\right)^{D} =$$
$$= \lambda(\boldsymbol{\sigma} - \mathbf{X})^{D} : \mathbf{b}(\boldsymbol{\sigma} - \mathbf{X})^{D} \geq 0 \tag{5.10}$$

The scalar reinforcement parameter æis unusually defined by the relation

$$\dot{æ} = \chi(\boldsymbol{\sigma}, \mathbf{e}'') \tag{5.11}$$

Here, χ is a non-negative continuous function which is homogeneous of the first order with respect to the second argument. Since $\partial\Re/\partial æ > 0$, the following inequality holds:

$$\dot{æ}\frac{\partial \mathfrak{K}}{\partial æ} \geq 0 \tag{5.12}$$

The parameter λ in relations (5.4) can be represented in explicit form. To do this, it is necessary to differentiate the right-hand side of equality (5.2) with respect to time and use expressions (5.4), (5.5) and (5.11). We then obtain

$$\lambda = \frac{\overset{\nabla}{f}}{\Delta}; \quad \Delta = \chi\frac{\partial \mathfrak{K}}{\partial æ} + \frac{1}{\rho F_0}\mathbf{d}^{-1} f_{\boldsymbol{\sigma}} : f_{\boldsymbol{\sigma}} - (1 - \varsigma) f_h : \dot{\mathbf{h}} \tag{5.13}$$

The function $\overset{\nabla}{f}$ is defined by formula (5.6).

Note that the quantity Δ is strictly positive. This follows from relation (5.12), the positive definiteness of the tensor **d** and the definition of the tensors $\dot{\mathbf{D}}$ and $\dot{\mathbf{h}}$, and, also, from the representation of the loading function.

The relation[7]

$$f(\boldsymbol{\sigma}, \mathbf{X}, \mathbf{H}) = (\mathbf{Z} - \mathbf{X}) : (\mathbf{Z} - \mathbf{X}) - r; \quad \mathbf{Z} = \left(\mathbf{H} \cdot \boldsymbol{\sigma}^{D} \cdot \mathbf{H}\right)^{D}$$

where r is a reinforcement parameter, has been taken to approximate of the loading function in describing the damaged state of a medium. We then obtain

$$\Delta = \chi\frac{\partial \mathfrak{K}}{\partial æ} + \frac{1}{\rho F_0} f_{\boldsymbol{\sigma}} : \mathbf{d}^{-1} f_{\boldsymbol{\sigma}} - \frac{\partial f}{\partial \mathbf{H}} : \dot{\mathbf{H}}, \quad \frac{\partial f}{\partial \mathbf{H}} = 2\left((\mathbf{Z} - \mathbf{X})^{D} \cdot \mathbf{H} \cdot \boldsymbol{\sigma}^{D} + \boldsymbol{\sigma}^{D} \cdot \mathbf{H} \cdot (\mathbf{Z} - \mathbf{X})^{D}\right) \tag{5.14}$$

The last term in the first relation of (5.14) can take both positive and negative values. It is therefore unclear whether the right-hand side of relation (5.14) is always positive or whether it vanishes for a certain set of values of the state parameters.

Remark. It should be possible by following the approach proposed earlier [7,11] to add a term of the form $-\omega\mathbf{BX}$ to the right-hand side of relation (5.8), where $\omega = \omega(T)$ is a certain function and $\mathbf{B} = \mathbf{B}(\mathbf{h})$ is a symmetric fourth rank tensor. In this case, the representation

$$\lambda = \frac{\overset{\nabla}{f}}{\tilde{\Delta}}, \quad \tilde{\Delta} = \Delta + \omega\frac{\partial f}{\partial \mathbf{X}} : \mathbf{BX}$$

holds for the parameter λ. The second term in the last relation can take both positive and negative values, no matter what the function ω and the tensor **B** are. It is impossible to prove that the right-hand side of this relation will be certainly positive without additional assumptions.

The following results were obtained in Sections 4 and 5.

In the case of media for which the loading function is convex with respect to the set of all the variables, the constitutive relations (4.6) follow from the normality principle and satisfy the condition $Q \geq 0$. In this case, it is not necessary to impose any additional conditions on the free energy (the Gibbs potential).

In the case of media for which the loading function is only convex with respect to the set of variables σ^D, **X**, relation (4.9) with the condition $\partial \mathbf{f}_D/\partial \mathbf{e}'' \neq 0$ or relation (4.11) with the condition $\partial \mathbf{f}_D/\partial \mathbf{e}'' = 0$, together with formulae (5.4) - (5.7) and (5.10) for determining the tensors $\mathbf{e}''$, $\dot{\mathbf{X}}$ and the quantity $\dot{æ}$ give an approximation of the constitutive relations for which conditions (2.3), (2.7), (5.9) and (5.12) hold and this means that the condition $Q \geq 0$ also holds.

6. Differentiation of the equation with respect to the governing parameters

The constitutive relations obtained above can be transformed and two equations can be obtained. The first of these is a tensor equation. It relates the rate of change of the stress tensor to the strain rate tensor and the tensor $\dot{\mathbf{h}}$, and is a generalization of Hooke's law. The second equation describes the heat transfer process. It is written in the form of a parabolic-type quasilinear operator which acts on the function $T = T(t, \mathbf{x})$.

In order to obtain the first equation, we differentiate the first equation of (2.6) with respect to time. We add the resulting equation to equality (2.1), taking account of the relation $d\varepsilon/dt = \mathbf{e}'$, and we then obtain

$$\mathbf{e} = \mathbf{e}'' + \mathbf{a}\dot{\boldsymbol{\sigma}} - \boldsymbol{\alpha}\dot{\theta} - \dot{\boldsymbol{\varepsilon}}_0 + \dot{\mathbf{a}}\boldsymbol{\sigma} - \dot{\boldsymbol{\alpha}}\theta; \quad \dot{\boldsymbol{\varepsilon}}_0 = \frac{\partial \boldsymbol{\varepsilon}_0}{\partial \mathbf{h}} : \dot{\mathbf{h}}, \quad \dot{\mathbf{a}} = \frac{\partial \mathbf{a}}{\partial \mathbf{h}} : \dot{\mathbf{h}}, \quad \dot{\boldsymbol{\alpha}} = \frac{\partial \boldsymbol{\alpha}}{\partial \mathbf{h}} : \dot{\mathbf{h}} \tag{6.1}$$

We now eliminate the tensor $\mathbf{e}''$ from Eq. (6.1), taking account of relations (5.4)–(5.7), (5.13) and then obtain

$$\dot{\boldsymbol{\sigma}} = (\mathbf{c} - \mathcal{K}\mathbf{c}f_{\boldsymbol{\sigma}} \otimes \mathbf{c}f_{\boldsymbol{\sigma}})\left(\mathbf{e} + \dot{\boldsymbol{\varepsilon}}_0 - \dot{\mathbf{a}}\boldsymbol{\sigma} - \frac{f_{\sigma}}{\Delta}(\dot{f} - f_{\sigma} : \dot{\boldsymbol{\sigma}}) + \boldsymbol{\alpha}\dot{\theta} + \dot{\boldsymbol{\alpha}}\theta\right); \quad \mathcal{K} = \frac{1}{(\mathbf{c}f_{\boldsymbol{\sigma}} : f_{\boldsymbol{\sigma}} + \Delta)} \tag{6.2}$$

In the case when the tensors $\mathbf{a}$ and $\boldsymbol{\alpha}$ are constants and $\varepsilon_0 = 0$, the known relation which holds for small deformations:

$$\mathbf{e} = \mathbf{e}'' + \mathbf{a}\dot{\boldsymbol{\sigma}} - \boldsymbol{\alpha}\dot{\theta}$$

follows from equality (6.1).

In the case of isothermal processes when $\dot{\varepsilon}_0 = \dot{\mathbf{a}} = \dot{\alpha} = 0$, expression (6.2) is transformed into the relation[12]

$$\dot{\boldsymbol{\sigma}} = \left(\mathbf{c} - \frac{\mathbf{c}f_{\boldsymbol{\sigma}} \otimes \mathbf{c}f_{\boldsymbol{\sigma}}}{\mathbf{c}f_{\boldsymbol{\sigma}} : f_{\boldsymbol{\sigma}} + \Delta}\right)\mathbf{e}$$

Note that relation (6.1) is a generalization of the Prandtl–Reisz equation for models of media in which non-isothermal processes and the accumulation of damage occur.

We now derive the second equation using relation (2.4). The heat flux vector $\mathbf{q}$ appears on the right-hand side of this relation and it is necessary to define a law which relates the vector $\mathbf{q}$ to the temperature gradient. Note that, in the undamaged state when the medium is isotropic, the relation between $\mathbf{q}$ and $\nabla\theta$ is given by Fourier's law

$$\mathbf{q} = -\pi\nabla\theta \tag{6.3}$$

Here, $\pi \geq 0$ is the thermal conductivity.

In the damaged state, the medium becomes anisotropic, and the vectors $\mathbf{q}$ and ∇T are now non-collinear. The relation between them is given using a symmetric, positive definite tensor $\boldsymbol{\Lambda}$, the heat conduction tensor:

$$\mathbf{q} = -\boldsymbol{\Lambda}\nabla\theta \tag{6.4}$$

We shall assume that, in describing the damaged state of the medium, the tensor $\boldsymbol{\Lambda}$ is represented in the form

$$\boldsymbol{\Lambda} = \pi q_k \mathbf{h}^k$$

Here, $q_k = q_k(\mathbf{h}) \geq 0$, $(k = 0,1,2)$ are functions satisfying the condition

$$\sum q_k(\mathbf{I}) = 1$$

It follows from the last equality that, when $\mathbf{h} = \mathbf{I}$, relations (6.3) and (6.4) are identical.

We will now consider the system of variables ε, T, $\mathbf{X}$, æ, $\mathbf{D}$ and transform the right-hand side of Eq. (2.4), taking relation (5.10) into account. We obtain an expression for the entropy from relations (2.5) and (3.1)

$$s = -\left(s_0 + c_{\boldsymbol{\varepsilon}}\theta + \frac{1}{\rho}\boldsymbol{\beta} : \boldsymbol{\varepsilon}\right)$$

Substituting the right-hand side of the last equality into Eq. (2.4) instead of s, we arrive at the equation

$$b_{\boldsymbol{\varepsilon}}\dot{\theta} = T_0\left(\rho\left(\dot{s}_0 + \dot{c}_{\boldsymbol{\varepsilon}}\theta\right) + (\boldsymbol{\beta} : \boldsymbol{\varepsilon})^{\bullet}\right) + \nabla \cdot (\boldsymbol{\Lambda}\theta) - \mathcal{Q} \tag{6.5}$$

For the system of variable σ, T, $\mathbf{X}$, æ, $\mathbf{D}$, taking account of the representation of the Gibbs potential (3.19) and relation (2.6), we similarly obtain

$$s = -\left(s_0 + c_{\boldsymbol{\sigma}}\theta + \frac{1}{\rho}\boldsymbol{\alpha} : (\boldsymbol{\sigma} - \boldsymbol{\sigma}_0)\right)$$

In the same way as above, after some reduction we arrive at the equation

$$b_{\boldsymbol{\sigma}}\dot{\theta} = T_0\left(\rho\left(\dot{s}_0 + \dot{c}_{\boldsymbol{\sigma}}\theta\right) + (\boldsymbol{\alpha} : (\boldsymbol{\sigma} - \boldsymbol{\sigma}_0))^{\bullet}\right) + \nabla \cdot (\boldsymbol{\Lambda}\theta) - \mathcal{Q} \tag{6.6}$$

In Eqs (6.5) and (6.6)

$$\mathcal{Q} = \frac{\overset{\nabla}{f} - f_T\dot{T}}{\Delta}f_{\boldsymbol{\sigma}} : (\boldsymbol{\sigma} - \mathbf{X}) - \rho\left(\frac{\partial\mathcal{F}}{\partial\mathbf{D}} : \dot{\mathbf{D}} + \frac{\partial\mathcal{F}}{\partial æ} : \dot{æ}\right), \quad b_{\boldsymbol{\varepsilon}} = \rho\tilde{c}_{\boldsymbol{\varepsilon}} + k_T, \quad b_{\boldsymbol{\sigma}} = \rho\tilde{c}_{\boldsymbol{\sigma}} + k_T, \quad k_T = \frac{\mathfrak{K}\partial\mathfrak{K}}{\Delta\,\partial T}$$

The partial derivatives in the operator Q are taken when $\boldsymbol{\varepsilon} = \text{const}$ (Eq. 6.5) and when $\boldsymbol{\sigma} = \text{const}$ (Eq. (6.6)).

In order for Eqs (6.5) and (6.6) to be equations of the parabolic type, the conditions

$$b_{\boldsymbol{\varepsilon}} > 0, \quad b_{\boldsymbol{\sigma}} > 0$$

must be satisfied.

If the Euler equations of motion are added to Eqs (6.2), (6.6), (5.4), (5.12) and (4.7), we obtain a system of partial differential equations in the variables **v**, σ, θ, **X**, æ **D** which will simulate the motion of the media considered.

More precisely, suppose there is a body occupying the domain $\Omega \subset \mathbb{R}^3$. Then, it is possible to formulate a mixed problem for this system of differential equations if the values of the quantities **v**, σ, θ, **X**, æ **D** are given in the domain Ω at a certain instant $t = t_0$ and, when $t \geq t_0$, the surface forces $\mathbf{R} = \mathbf{R}(\mathbf{x})$ are given on the part Γ_σ of the boundary $\partial\Omega$ of the domain Ω and the velocity $\mathbf{v}_0 = \mathbf{v}_0(\mathbf{x})$ on the complementary part $\partial\Omega/\Gamma_\sigma$ of the boundary. Similarly, it is possible to derive a system of partial differential equations in the variables **v**, ε, θ, **X**, æ **D**.

Taking account of the symmetry and positive definiteness of the tensor **c**, it can be shown that the resulting system of differential equations is hyperbolic in the isothermal case. This fact is a necessary condition for the rigorous mathematical proof of theorems on the existence and uniqueness of the the solution of this mixed problem and other problems which arise when modelling simulation of evolutionary processes in reinforce elastoplastic media, taking account of damage kinetics.

The proposed method of approximating of the constitutive relations can be extended to a fairly wide class of materials used in machine construction and which are isotropic in the undamaged state.

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